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EE 364 - Fall 2026

# Week 3
# Supplemental Notes



3 types of proof

1. Direct : $P \longrightarrow Q$

   assume $P$, derive $Q$

2. Indirect : "reductio ad absurdum" (reduce to absurdity)

   assume $P \wedge \sim Q$, derive $C \wedge \sim C$

   $$P \longrightarrow Q \overset{M.I.}{=} \sim P \vee Q = \sim(P \wedge \sim Q)$$

   $$\therefore \sim(P \longrightarrow Q) = \boxed{P \wedge \sim Q}$$

<u>Thrm</u>: $(P \longrightarrow (Q \,\&\, \sim Q)) \longrightarrow \sim P.$

| $P$ | $Q$ | $P$ | $\longrightarrow$ | $(Q \wedge \sim Q)$ | $\Longrightarrow$ | $\sim P$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 |

$\therefore$ Valid

Ex: Thrm: $\forall \varepsilon > 0 \quad a \le b + \varepsilon \longrightarrow a \le b.$

Prf: Suppose not. i.e. $\forall \varepsilon > 0 : (a \le b + \varepsilon) \wedge (a > b)$

$$a = \frac{a}{2} + \frac{a}{2} \overset{\text{Assum.}}{>} \frac{a}{2} + \frac{b}{2} = b - \frac{b}{2} + \frac{a}{2} = b + \frac{a-b}{2}$$

$\varepsilon' > 0$ since:

$a > b \longleftrightarrow a - b > 0$

$$= b + \varepsilon' \wedge \varepsilon' > 0$$

$\therefore \frac{a-b}{2} > 0$

$\therefore \varepsilon' > 0$

$\therefore \quad a > b + \varepsilon' \quad w/ \ \varepsilon' > 0$

$\rightarrow\leftarrow$ contradiction: since $a \le b + \varepsilon \quad \forall \varepsilon > 0$

Thrm: $Q \wedge \sim Q \longrightarrow P$

(true AND false imply anything)

# CUT CAT

- acronym (mnemonic) for a probability space

## ① CUT

- definition of a sigma-algebra

Space $X$ ("Sample space" $\Omega$)

$2^X = \{A : A \subset X\}$ - power set
- set of all subsets.

$|2^X| = 2^n$

# elements in power set.

$\therefore \; x \in A \subset X \in \mathcal{Q} \subset 2^X$.

if $\mathcal{Q}$ is a subcollection of $2^X$ (that contains $X$)

set of sets ("class")

**Ex:** $X = \{1,2,3\}$ $\quad 2^X = \cdots$ , note: $|2^X| = 2^3 = 8$ ($2^8$ possible set coll.)

**Defn:** $\mathcal{Q} \subset 2^X$ is a sigma-algebra (S.A. or $\sigma$-algebra)
iff $\mathcal{Q}$ is CUT:

C: closed under complement.

$$A \in \mathcal{Q} \Longrightarrow A^c \in \mathcal{Q}$$

U: closed under countable unions

$$A_1 \in \mathcal{Q},\; A_2 \in \mathcal{Q}, \ldots \Longrightarrow \bigcup_k A_k \in \mathcal{Q}$$

T: total set $X$

$$X \in \mathcal{Q}$$

# IMAC format

I: Issue

- associative memory ("spot the issue")

M: Math rule

- rote memorization

A: Apply math to facts (Analysis)

- pattern matching
- element because fact.

C: Conclusion

- deductive logic

# IMAC example: Finite sigma-algebra

Note: HW1/2

Sample space $X$ contains four elements $X = \{a, b, c, d\}$. So its power set or set of subsets $2^X$ contains $2^4$ elements. Thus the power set $2^X$ gives rise to $2^{16}$ subcollections of sets. Suppose that $A \subset 2^X$ is one of these $2^{16}$ set subcollections of $2^X$ and that $A$ contains the following 7 sets:

$$A = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,c,d\}, \{b,c,d\}, X\}.$$

Can we define a probability space $(X, A, P)$ using the set collection $A$ if $P$ is a probability measure?

(use IMAC format to answer)

Soln:

$\underline{\text{I}}$: sigma-algebra (s.a.)

$\underline{\text{M}}$: $\mathcal{Q} \subset 2^X$ is a sigma-algebra

iff $\mathcal{Q}$ is $\underline{\text{CUT}}$:

$$\begin{cases} C: & A \in \mathcal{Q} \implies A^c \in \mathcal{Q} \\ U: & A_1 \in \mathcal{Q},\ A_2 \in \mathcal{Q}, \dots \implies \bigcup_{k=1}^{\infty} A_k \in \mathcal{Q} \\ T: & X \in \mathcal{Q} \end{cases}$$

$\underline{\text{A}}$:

$\underset{\sim}{C}$:

$\{a\}^c = \{b,c,d\} \in \mathcal{Q}$

$\{b\}^c = \{a,c,d\} \in \mathcal{Q}$.

$\{a,b\}^c = \{c,d\} \notin \mathcal{Q}$

$\{b,c,d\}^c = \{a\} \in \mathcal{Q}$

$\{a,c,d\}^c = \{b\} \in \mathcal{Q}$

$\phi^c = X \in \mathcal{Q}$, $X^c = \phi \in \mathcal{Q}$

$\therefore$ $\mathcal{Q}$ is not closed under complement

because $\{a, b\}^c \notin \mathcal{Q}$

U: $\{a\} \cup \{b\} = \{a, b\} \in \mathcal{Q}$

$\{a\} \cup \{a, b\} = \{a, b\} \in \mathcal{Q}$

$\{a\} \cup \{a, c, d\} = \{a, c, d\} \in \mathcal{Q}$

$\{a\} \cup \{b, c, d\} = X \in \mathcal{Q}$

$\{b\} \cup \{a, b\} = \{a, b\} \in \mathcal{Q}$

$\{b\} \cup \{a, c, d\} = X \in \mathcal{Q}$

$\{b\} \cup \{b, c, d\} = \{b, c, d\} \in \mathcal{Q}$

$\emptyset \cup S = S \in \mathcal{Q}$ if $S \in \mathcal{Q}$

$X \cup S = X \in \mathcal{Q}$ if $S \in \mathcal{Q}$

$\therefore$ $\mathcal{Q}$ is closed under union.

T: $X \in \mathcal{Q}$.

$\therefore$ $\mathcal{Q}$ is ~~C~~UT

$\therefore$ $\mathcal{Q}$ is not a sigma-algebra.

C: **NO.**

$(X, A, P)$ cannot be a probability space

because $A$ is not a sigma-algebra.

# Discrete Sample Spaces

Finite or $\underbrace{\text{denumerable}}$ outcomes

→ can put into 1-to-1 correspondance w/ $\mathbb{N}$

Ex: $\Omega = \{ a_1, a_2, \dots, a_n \}$ finite sample space

Consider "equally likely outcomes"

$$P[\{a_i\}] = \frac{1}{n} \quad \forall i = 1, \dots, n$$

If event consists of k distinct outcomes:

$$P[B] = P[\{a_1', a_2', \dots, a_k'\}] = P[\{a_1'\}] + \dots + P[\{a_k'\}]$$

$$= \frac{k}{n}$$

Ex: 3 coin flips : 8 outcomes

$$\Omega = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$$

Assume equiprobable : $P[xxx] = \frac{1}{8}$

$$P[2 \text{ heads}] = P[HHT \text{ or } HTH \text{ or } THH] = \frac{3}{8}$$

# Counting methods and Conbinatorics

In finite sample space w/ equiprobable outcomes.

$$P[\text{event}] = \frac{\text{distinct \# of outcomes that match event}}{\text{total \# distinct outcomes}}$$

Ex: multiple choice exam, 10 questions, 4 options each

$$\# \text{ outcomes} = \underbrace{4 \times 4 \times \dots \times 4}_{10} = 2^{20} \approx 1 \text{ million}$$

Computer Problem: Birthday Paradox

Q: How large of group of k people to have prob. at least 0.5 of having at least two people with same birthday

(assume: 365 days/year, independent birthdays, equally likely per day)

Let $A_k$ = event at <u>least</u> 2 people in group w/ k share a birthday.

define $q_k = P[\underbrace{A_k^c}_{\text{no common birthday in group of } k.}]$

$q(1) = 1$

$q(2) = q(1) \cdot \frac{364}{365} = \frac{\cancel{365}}{365} \cdot \frac{364}{\cancel{365}} = \frac{364}{365}$

$q(3) = q(2) \cdot \frac{363}{365} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}$

$\vdots$

$q(k) = q(k-1) \cdot \frac{365-k+1}{365} = \frac{365 \cdot 364 \cdot (365-k+1)}{365^k} = \frac{(365)_k}{365^k}$

"365 fall k"

Computer: $q(22) = 0.524$

$q(23) = 0.493$ $\therefore k \geq 23$ people.

# Random sampling (ordered)

- with replacement
  - choose k from n distinct elements.
  - select 1 at a time and "replace" after each selection

    $\therefore$ duplicate samples possible

  \# outcomes: ordered k-tuple w/ n-options for each position

  $$= \underbrace{n \cdot n \cdots n}_{k} = n^k$$

  Ex: multiple choice exam w/ 4-options per question.

- without replacement
  - choose k from n distinct elements
  - select 1 at a time & "remove" after each selecting

    $\therefore$ no duplicate samples possible

  $n_1 = n$

  $n_2 = n-1$

  $n_3 = n-2$

  $\vdots$

  $n_k = n-k+1$

  $\therefore$ \# outcomes $= n \cdot (n-1)(n-2) \cdots (n-k+1)$

  (note $k \leq n$)

  Ex: Drawing balls \#1-60 from urn, don't replace

  Special case: permutation.

  ↳ Sampling w/o replacement, $k = n$

  \# outcomes $= n \cdot (n-1)(n-2) \cdots (2)(1)$

  $= \boxed{n!}$

**Defn:** Permutation = Self-Bijection

- $f: S \to S$
- $f:$ 1-to-1 (injection)
- $f:$ onto (surjection)

## Fundamental Rule of counting

$$\text{arrangement} = \boxed{n_1 \,|\, n_2 \,|\, \cdots \,|\, n_k}$$

$$\therefore \; |\text{arrangements}| = \prod_{k=1}^{n} n_k$$

$$\left( = n^k \text{ if } n_k = n \;\; \forall k \right)$$

$P(n,k)$ = # permutations of length $k$ on $n$ elements

$$n \cdot (n-1) \cdots (n-(k+1)) \cdot \frac{(n-k)!}{(n-k)!}$$

$$= \frac{n!}{(n-k)!} \quad \text{for } k \leq n.$$

**Defn:** $n! = n\,(n-1) \cdots 3 \cdot 2 \cdot 1$ for $n \in \{0, 1, 2, \ldots\}$.

$(0! = 1)$

**Fact:** (Stirling's Approximation)

$$\ln n! \approx n \cdot \ln n - n$$

$$\approx n \cdot \ln n$$

Ex: $n! < n^n$, for $n > 2$

$\therefore \quad \ln n! < \ln n^n = n \cdot \ln n$

$n^{\text{th}}$ term

$e^n = \sum_{k=0}^{\infty} \frac{n^k}{k!} > \frac{n^n}{n!}$ ⟵ since $\sum_{k=0}^{\infty} \frac{n^k}{n!} = \frac{n^0}{0!} + \frac{n}{1!} + \frac{n^2}{2!} + \dots + \frac{n^n}{n!} + \dots$

$\therefore \quad n! > \frac{n^n}{e^n}$

$\therefore \quad \ln n! > n \cdot \ln n - n$

$\therefore \quad n \cdot \ln n - n < \ln n! < n \cdot \ln n$

$\therefore \quad 1 - \frac{1}{\ln n} < \frac{\ln n!}{n \cdot \ln n} < 1$

$\therefore \quad \lim_{n \to \infty} \frac{\ln n!}{n \cdot \ln n} = 1$ $\qquad \therefore \quad \ln n! \sim n \cdot \ln n$

Note: $\ln n! = \ln \prod_{k=1}^{n} k = \sum_{k=1}^{n} \ln k \sim \int_{x=1}^{n} \ln x \, dx$

pick $u = \ln x \qquad dv = dx$

$du = \frac{1}{x} \qquad v = x$

$$= x \cdot \ln x \Big|_{x=1}^{x=n} - x \Big|_{x=1}^{x=n} = n \cdot \ln n - n + 1 \sim n \cdot \ln n - n$$

# Integration by parts

**Thrm:** $$\int u \cdot dv = uv - \int v \cdot du$$

**Prf:** $$(uv)' = u'v + uv' = du\, v + u\, dv$$

$$\therefore \quad u\, dv = (uv)' - v \cdot du$$

$$\therefore \quad \int u\, dv = uv - \int v \cdot du$$

**Thrm:** (Stirling's approximation) $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

# Unordered Sampling

order doesn't matter, i.e., $(a, b) \sim (b, a)$

- without replacement
  - choose k from n w/o replacement
  - outcome order does <u>not</u> matter

<u>Ex</u>: "Standard lottery"

Draw 6 balls — "lottery numbers"

any order wins

$\therefore$ 60 balls choose 6 : $60 \cdot 59 \cdots 55 = \dfrac{60!}{54!}$ ← $\dfrac{n!}{(n-k)!}$

but each outcome has : $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ equivalent outcomes

$\therefore$ # outcomes $= \dfrac{60!}{54!\,6!}$

<u>Defn</u>: $\dfrac{n!}{(n-k)!\,k!} = \binom{n}{k}$ is the <u>binomial coefficient</u>

- with replacement
  - choose k from n w/ replacement.
  - outcome order does <u>not</u> matter.

<u>Ex</u>: "Pizza toppings"

k → 4 toppings
+
n → 5 choices
(duplicate OK)

| A | B | C | D | |
|---|---|---|---|---|
| XX | | X | XX | ← outcome #1 |
| X | XX | X | X | ← outcome #2 |

Shorthand: XX | | X | XX   #1   5 "X" + 3 "|"

X | XX | X | X   #2   "stars and bars"

$\therefore$ every outcome unordered w/ replacement is equivalent to a unique ordered sample of "X" and "|"
($k$ "X" and $n-1$ "|")

$\therefore$ # distinct outcomes : sample w/o replacement from urn w/ $k$-black balls and $(n-1)$ white balls.

( $\therefore$ total $= n-1+k$ balls )

Ex: (cont) How many distinct permutations (ordered) of $k$ black balls and $n-1$ white balls?

Consider balls # $1, 2, \dots, n-1+k$ (no color)

then choose $k$, put "X" there. "|" else.

$$\frac{(n-1+k)!}{k!\,(n-1)!} \quad \binom{n-1+k}{k} = \binom{n-1+k}{n-1} \quad \frac{(n-1+k)!}{(n-1)!\,k!}$$

choose $k$ for "X" pos. choose $n-1$ for "|" pos.

## Counting method summary

| Ordered \ Replacement | with | without |
|---|---|---|
| yes | $n^k$<br>Ex: # ways to answer multiple choice test | $n(n-1)\cdots(n-(k-1))$<br>Ex: How many ways to rank $k$ students in class with $n$ students |
| no | $\frac{(n-1+k)!}{(n-1)!\,k!}$<br>Ex: # different pizzas w/ $k$-toppings (duplicate OK) | $\frac{n!}{k!\,(n-k)!}$<br>Ex: # ways to draw $k$ "lottery numbers" |

$$\binom{n-1+k}{n-1} = \binom{n-1+k}{k}$$

$$\binom{n}{k}$$

# Binomial Coefficients

$P(n,k)$ = # permutations of length $k$ on $n$ elements

$$= \frac{n!}{(n-k)!} \quad \text{for } k \le n.$$

$C(n,k)$ = # combinations of length $k$ on $n$ elements

$$= \binom{n}{k} = \frac{n!}{(n-k)!\,k!} \quad \text{for } k \le n.$$

Note: $\binom{n}{0} = 1 \longleftrightarrow \emptyset$

$\binom{n}{n} = 1 \longleftrightarrow \Omega$

## Pascal's triangle

$\binom{3}{1}$ $\qquad$ $\binom{3}{2}$ $\qquad$ $\binom{4}{2}$

0: 1

1: 1 1

2: 1 2 1

3: 1 3 3 1

4: 1 4 6 4 1

5: 1 5 10 10 5 1

⋮

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = \dots$

$(x+y)^5 = \dots$

note: # ways of choosing

__ __ __

w/ 2 "x" and 1 "y"

just a hypothesis now
proof is binomial theorem

for $\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$

**Thrm:** (Pascal's Formula) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

**Prf:** $\binom{n-1}{k-1} + \binom{n-1}{k} = \dfrac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} + \dfrac{(n-1)!}{k!\,(n-1-k)!}$

$$= \frac{k}{k}\cdot\frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}\cdot\frac{n-k}{n-k}$$

$$= \frac{(n-1)!\,k}{k!\,(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,(n-k)!}$$

$$= \frac{(n-1)!\,(n-\cancel{k+k})}{k!\,(n-k)!} = \frac{(n-1)!\,n}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k}$$

QED.

**Review:** how to change the <u>INDEX of summation</u>

**Show:** $\displaystyle\sum_{k=1}^{n} a_{k-1} = \sum_{j=0}^{n-1} a_j$

**Technique:** $\displaystyle\sum_{k=1}^{n} a_{k-1} = \sum_{k=1}^{k=n} a_{k-1}$ put $j = k-1$ $\therefore k = j+1$

$$= \sum_{j+1=1}^{j+1=n} a_j$$

$$= \sum_{j=0}^{j=n-1} a_j$$

$$= \sum_{j=0}^{n-1} a_j$$

QED.

★★★

**Thrm:** (Binomial Theorem)

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.$$

**Prf:** (by induction on $n = 1, 2, 3, \ldots$)

**Basis:** $n = 1$

$$\therefore (p+q)^1 = p+q.$$

$$\sum_{k=0}^{1} \binom{1}{k} p^k q^{n-k} = \binom{1}{0} p^0 q^1 + \binom{1}{1} p^1 q^0$$

$$= p+q.$$

QED (basis)

**Induction Step** Induction hypothesis: $(p+q)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k}$

$$\therefore (p+q)^n = (p+q)(p+q)^{n-1}$$

$$\overset{IH}{=} (p+q) \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} p^{k+1} q^{n-k-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{n-k}$$

$$= \underbrace{\sum_{j=1}^{n} \binom{n-1}{j-1} p^j q^{n-j}}_{j=k+1 \quad \therefore k=j-1} + \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{n-j}}_{j=k}$$

$$= \left[ \underbrace{\binom{n-1}{n-1}}_{=1} \underbrace{p^n q^{n-n}}_{p^n} + \sum_{j=1}^{n-1} \binom{n-1}{j-1} p^j q^{n-j} \right] +$$

$$\left[ \sum_{j=1}^{n-1} \binom{n-1}{j} p^j q^{n-j} + \underbrace{\binom{n-1}{0}}_{=1} \underbrace{p^0 q^{n-0}}_{q^n} \right]$$

$$= p^n + \sum_{j=1}^{n-1} \binom{n-1}{j-1} p^j q^{n-j} + \sum_{j=1}^{n-1} \binom{n-1}{j} p^j q^{n-j} + q^n.$$

$$= p^n + \sum_{j=1}^{n-1} \left[ \underbrace{\binom{n-1}{j-1} + \binom{n-1}{j}}_{=\binom{n}{j}} \right] p^j q^{n-j} + q^n$$

$$= p^n + \sum_{j=1}^{n-1} \binom{n}{j} p^j q^{n-j} + q^n$$

$$= \binom{n}{n} p^n q^{n-n} + \sum_{j=1}^{n-1} \binom{n}{j} p^j q^{n-j} + \binom{n}{0} p^0 q^{n-0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

QED.

## 3 Corollaries

1. $p = q = 1$ $\quad \therefore \sum_{k=0}^{n} \binom{n}{k} = 2^n$

since $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

2. $p + q = 1$ w/ $p \geq 0,\ q \geq 0$ $\quad (\therefore q = 1 - p)$

$$\therefore \sum_{k=0}^{n} P(X=k) \overset{\text{binomial}}{=} \sum_{k=0}^{n} b(n,k,p) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$$\overset{BT}{=} (p+q)^n = 1^n = 1 \qquad \left(\text{w/ } b(n,k,p) = \binom{n}{k} p^k q^{n-k}\right)$$

$\therefore$ Binomial $b(n,k,p)$ is a probability density

3. $p = -1$ and $q = +1$ $\quad \therefore \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$$